

NOTE

ON MINIMIZING SYMMETRIC SET FUNCTIONS

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Mader proved that every loopless undirected graph contains a pair (u, v) of nodes such that the star of v is a minimum cut separating u and v . Nagamochi and Ibaraki showed that the last two nodes of a “max-back order” form such a pair and used this fact to develop an elegant min-cut algorithm. M. Queyranne extended this approach to minimize symmetric submodular functions. With the help of a short and simple proof, here we show that the same algorithm works for an even more general class of set functions.

Main section

Let V be a finite set. A value $d(\{S, T\})$ is given for every unordered pair of disjoint subsets S, T of V . For convenience, function d is called a *map* on V , even if it is actually defined on a subset of $2^V \times 2^V$. We also rely on the shorthand $d(S, T) = d(\{S, T\})$ and leave the fact that $d(S, T) = d(T, S)$ as understood. Function d is called *monotone* if $d(S, T') \leq d(S, T)$ for any S, T disjoint and $T' \subseteq T$. Finally, d is *consistent* if $d(A, W \cup B) \geq d(B, W \cup A)$ whenever A, B, W are disjoint sets such that $d(A, W) \geq d(B, W)$. As an example, when $G = (V, E)$ is an undirected graph, then $d(S, T) = |\{st \in E : s \in S, t \in T\}|$ for any disjoint sets $S, T \subseteq V$, is a monotone and consistent map

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on V . A subset S of V is said *nontrivial* when $\emptyset \neq S \neq V$. We give an efficient algorithm to solve the following problem (*minimum bipartition problem*):

Given a finite set V and a monotone and consistent map d on V ,
find a nontrivial subset S of V for which $d(S, V \setminus S)$ is minimum.

A *max-back order* for (V, d) is an ordering v_1, v_2, \dots, v_n of the elements in V such that

$$d(v_i, \{v_1, \dots, v_{i-1}\}) \geq d(v_j, \{v_1, \dots, v_{i-1}\}) \quad \text{for } 2 \leq i < j \leq n$$

Let s and t be two elements of V . An *st-set* is a subset S of V with $|S \cap \{s, t\}| = 1$.

An ordered pair (s, t) of elements of V is *good* if $d(\{t\}, V \setminus \{t\}) \leq d(S, V \setminus S)$ holds for every *st-set* S . Before the end of this section we will prove the following lemma:

Lemma 1. *Let v_1, \dots, v_n be a max-back order for (V, d) . Then (v_{n-1}, v_n) is good for (V, d) .*

Lemma 1 gives an efficient procedure, called *a_Good_Pair*, to find a good pair. When (s, t) is a good pair two cases are possible: either $\{t\}$ is an optimal solution to our problem, or no optimal solution S to the problem is an *st-set*. This motivates the following definitions: Let s and t be any two elements of V . Consider *identifying* s and t into a single new element v_{st} thus obtaining a new set $V_{st} = V \setminus \{s, t\} \cup \{v_{st}\}$. Now, to reconsider a subset X of V_{st} as a subset of V , we define $\langle X \rangle = X$ if $v_{st} \notin X$ and $\langle X \rangle = X \setminus \{v_{st}\} \cup \{s, t\}$ if $v_{st} \in X$. When S and T are disjoint subsets of V_{st} then $\langle S \rangle$ and $\langle T \rangle$ are disjoint subsets of V and we define:

$$d_{st}(S, T) = d(\langle S \rangle, \langle T \rangle)$$

Note that, when d is a monotone and consistent map on V , then d_{st} is a monotone and consistent map on V_{st} . To conclude, the following algorithm solves the minimum bipartition problem.

Algorithm 1. MIN_BIPARTITION (V, d)

1. if $|V|=2$ then return either of the two nontrivial subsets of V ;
2. $(s, t) \leftarrow a_Good_Pair(V, d)$;
3. return the best set among $\{t\}$ and $\langle Min_Bipartition(V_{st}, d_{st}) \rangle$;

We have now enough motivation to prove Lemma 1.

Proof of Lemma 1. The lemma is true for $n=3$ since $d(v_2, v_1) \geq d(v_3, v_1)$ implies $d(\{v_1, v_3\}, v_2) \geq d(\{v_1, v_2\}, v_3)$ for d is consistent. Let \mathcal{S} be any $v_n v_{n-1}$ -set. We must show that:

$$(1) \quad d(\mathcal{S}, V \setminus \mathcal{S}) \geq d(\{v_n\}, V \setminus \{v_n\})$$

Clearly, $v_{v_1 v_2}, v_3, v_4, \dots, v_n$ is a max-back order for $(V_{v_1 v_2}, d_{v_1 v_2})$. Thus, either (1) follows by induction or \mathcal{S} is a $v_1 v_2$ -set. Since d is monotone, $v_1, v_{v_2 v_3}, v_4, \dots, v_n$ is max-back for $(V_{v_2 v_3}, d_{v_2 v_3})$ and either (1) follows or \mathcal{S} is a $v_2 v_3$ -set. Assume therefore that \mathcal{S} is both a $v_1 v_2$ -set and a $v_2 v_3$ -set. But then \mathcal{S} is not a $v_1 v_3$ -set and to derive (1) it suffices to show that $v_2, v_{v_1 v_3}, v_4, \dots, v_n$ is max-back for $(V_{v_1 v_3}, d_{v_1 v_3})$. Assume on the contrary $d_{v_1 v_3}(v_k, v_2) > d_{v_1 v_3}(v_{v_1 v_3}, v_2)$. However $d(v_2, v_1) \geq d(v_3, v_1)$ and $d(v_3, \{v_1, v_2\}) \geq d(v_k, \{v_1, v_2\})$ since v_1, \dots, v_n is max-back for (V, d) . Since d is monotone and consistent, we get $d(v_3, \{v_1, v_2\}) \geq d(v_k, \{v_1, v_2\}) \geq d(v_k, v_2) = d_{v_1 v_3}(v_k, v_2) > d_{v_1 v_3}(v_{v_1 v_3}, v_2) = d(\{v_1, v_3\}, v_2) \geq d(v_3, \{v_1, v_2\})$, a contradiction. ■

Some applications

A couple of observations and a list of applications will follow. In Application 1, Queyranne's important result on minimizing symmetric submodular functions is derived as a special case of our framework. The generalization is strict as shown in Applications 2 and 3.

Note that Algorithm 1 can also be used to solve maximization problems when $-d$ is a monotone and consistent map. In practice it follows that we can maximize $d'(S, V \setminus S)$ over the nontrivial subsets S of V whenever d' is a map on V with the following properties:

- (i) $d'(S, T') \geq d'(S, T)$ for any S, T disjoint and $T' \subseteq T$ - (reverse monotonicity);
- (ii) $d'(A, W \cup B) \geq d'(B, W \cup A)$ whenever A, B, W are disjoint sets such that $d'(A, W) \geq d'(B, W)$ - (consistency);

In contrast, maximizing $d(S, V \setminus S)$ for a generic monotone and consistent map d is an *NP*-complete problem since it contains as a special case the max-cut problem, which is known to be *NP*-complete [4].

Application 1 (symmetric submodular functions [9]).

Consider a finite set V and a real function f on 2^V . We are interested in finding a nontrivial subset of V which minimizes f . For this reason we consider an ordered pair (s, t) of elements of V to be *good* if $\{t\}$ is an *st*-set minimizing f . For any two disjoint subsets S, T of V let us define

$$d_f(S, T) = f(S) + f(T) - f(S \cup T)$$

If f is *symmetric* (that is, $f(S) = f(V \setminus S)$ for every subset S of V), then a pair is good with respect to f if and only if it is good with respect to d_f .

Note that d_f is consistent. Assume indeed A, B, W to be disjoint and such that $d_f(A, W) \geq d_f(B, W)$. This means $f(A) + f(W) - f(A \cup W) \geq f(B) + f(W) - f(B \cup W)$. But then $d_f(A, B \cup W) = f(A) + f(B \cup W) - f(A \cup B \cup W) \geq f(B) + f(A \cup W) - f(A \cup B \cup W) = d_f(B, A \cup W)$.

So we are interested in characterizing those f for which d_f is monotone, that is, $d_f(S, T_1) \leq d_f(S, T_1 \cup T_2)$ for any S, T_1, T_2 , all disjoint and non-empty. In terms of f this means, $f(S) + f(T_1) - f(S \cup T_1) \leq f(S) + f(T_1 \cup T_2) - f(S \cup T_1 \cup T_2)$, or equivalently, $f(S \cup T_1 \cup T_2) + f(T_1) \leq f(T_1 \cup T_2) + f(S \cup T_1)$. Hence d_f is monotone if and only if f satisfies the *submodular inequality* $f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$ for any sets A and B such that $A \setminus B, B \setminus A, A \cap B$ and $V \setminus A \setminus B$ are all non-empty.

In [8], Nagamochi and Ibaraki called such a function f *crossing submodular* and observed that the approach proposed by Queyranne in [9] to minimize symmetric submodular functions (where the submodular inequality has to hold for any sets A and B), was also valid for symmetric crossing submodular functions.

Algorithm 1 was first employed by Nagamochi and Ibaraki [7] to find minimum cuts in undirected graphs. A simple proof of the validity of Nagamochi and Ibaraki's min-cut algorithm had been obtained by Frank [2] and Stoer and Wagner [10], while Queyranne was deriving his important, but less simple, extension. Recently, in [3], Fujishige gave another short proof of the validity of Nagamochi and Ibaraki's min-cut algorithm and indicated how to employ his arguments to obtain a compact proof of Queyranne's result.

In the next application we show that our simple approach actually embraces an even broader class of problems.

Application 2 (short distance partitions).

Let G be a graph. A symmetric distance $\lambda(u, v)$ is given for every two nodes u, v . Assume we want to bipartition the node set V of G as to keep the maximum distance among two nodes on different sides of the partition as small as possible. Even if this problem can easily be solved directly, define $d(S, T) = \max\{\lambda(s, t) : s \in S, t \in T\}$. Note that d is a monotone and consistent

map in general. Consider the graph $(V, E) = (\{a, b, c, d\}, \{ab, bc, cd, da\})$ and for every $u, v \in V$ define the distance $\lambda(u, v)$ as the length of a shortest path between u and v . (Hence $\lambda(a, c) = \lambda(b, d) = 2$, and $\lambda(a, b) = \lambda(b, c) = \lambda(c, d) = \lambda(a, d) = 1$). The sets $S = \{a, c\}$ and $T = \{a, d\}$ show that the function f on 2^V defined by $f(S) = d(S, V \setminus S)$ for every $S \subseteq V$, is not crossing submodular in this special case.

Application 3 (critical cuts).

Let (G, w) be a weighted graph. Assume to be interested in those spanning trees T of G such that $\max\{w(e) : e \in T\}$ is as small as possible. Then it is natural to define the cost of a cut $\delta(S)$ as the minimum of $w(e)$ for $e \in \delta(S)$ and to search for a cut of maximum cost. This is clearly a bottleneck problem and admits a direct and simple solution.

Define $d(S, T) = \min\{w(e) : e \text{ has an endpoint in } S \text{ and the other in } T\}$. Note that $-d$ is a monotone and consistent map. This is indeed a reformulation of the above problem on short distance partitions (see Application 2). Hence we also have that the function f on 2^V defined by $f(S) = d(S, V \setminus S)$ for every $S \subseteq V$, is not crossing supermodular in general. (A function f is called *crossing supermodular* if $f(A \cap B) + f(A \cup B) \geq f(A) + f(B)$ for any sets A and B such that $A \setminus B, B \setminus A, A \cap B$ and $V \setminus A \setminus B$ are all non-empty).

Application 4 (minimum cuts in hypergraphs [5]).

Hypergraphs generalize graphs. When $G = (V, H)$ is an *hypergraph*, then the *hyperedges* in H are arbitrary subsets of the node set V . Thus a graph is an hypergraph in which every hyperedge has cardinality 2. Klimmek and Wagner [5] proposed a Nagamochi-Ibaraki type algorithm to find a minimum cut in an hypergraph. Indeed, the cut function of an hypergraph is symmetric and submodular [5, 9]. Consider the bottleneck version of this problem, that is, finding a cut which minimizes the maximum weight of an hyperedge belonging to it. Submodularity is lost but still we would have to deal with a monotone and consistent map.

Application 5 (partitions minimizing ambivalence).

Let G be a graph. Partition the node set V as $S \cup (V \setminus S)$ in such a way as to minimize the number of nodes with neighbors in both sides of the partition. This problem can be formulated as an hypergraph min cut problem (for every node v , we have an hyperedge h_v made of the neighbors of v in G). The problem hence falls in the framework of Stoer and Wagner [10], but also in that of Queyranne [9], or finally in our framework.

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